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Hoenders, B.J.; Pattanayak, D.N.

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Interaction of a moving charged particle with a spatially dispersive medium. II. Čerenkov and transition radiation*

B. J. Hoenders† and D. N. Pattanayak‡
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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In the preceding paper, we obtained expressions for the electromagnetic field generated by the interaction of a uniformly moving electron with a spatially dispersive half-space. One part of the field was identified with Čerenkov radiation and the other part with transition radiation. In this paper it is shown that the integrals involved in the Čerenkov part can be evaluated in closed form in terms of elementary functions, and we obtain three distinct threshold conditions for Čerenkov radiation instead of the usual one, as is the case in a spatially nondispersive medium. Furthermore, we obtain for each frequency component the asymptotic expansion of the transition radiation part of the field, valid at points that are far enough away from the path of the electron and from the boundary.

I. INTRODUCTION

In the preceding paper[1] (to be referred to as I), we derived expressions for the Čerenkov and transition radiation fields [Eq. (4.15) and Eqs. (4.22)–(4.24) of I] arising from the interaction of a uniformly moving electron passing through a spatially dispersive half-space. The constitutive relation of the medium was assumed to be of the form

$$\epsilon(\vec{k},\omega)=\epsilon_0(\omega)+\frac{\chi}{\vec{k}^2-\mu^2(\omega)}.$$

In such a medium the displacement vector $\vec{D}$ and the electric field vector $\vec{E}$ are found to be related by the formula

$$\vec{D}(\vec{r},\omega)=\epsilon_0(\omega)\vec{E}(\vec{r},\omega)+\int G(\vec{r},\vec{r}\,',\omega)\vec{E}(\vec{r}\,',\omega)d^3r'\,, \tag{1.1}$$

where

$$G(\vec{r},\vec{r}\,',\omega)=\exp(i\mu|\vec{r}-\vec{r}\,'|)/|\vec{r}-\vec{r}\,'|\,. \tag{1.2}$$

Equations (1.1) and (1.2) are the same as Eqs. (2.4) and (2.5) of I except that the volume of integration is now over the half-space. In this paper we will discuss the behavior of these fields. More specifically we will express, for each frequency component, the Čerenkov radiation field in terms of the modified Bessel functions of the third kind and obtain the asymptotic expansions valid for points at distances many wavelengths away from the path of the electron. We then discuss the threshold conditions for Čerenkov radiation inside the spatially dispersive medium.

We also obtain the asymptotic expansion for each frequency component of the transition radiation outside the spatially dispersive medium, valid at points that are at distances many wavelengths away from the path of the electron and the boundary.

II. ČERENKOV RADIATION

The particular solution inside the spatially dispersive half-space [as given by the Eqs. (4.15) of paper I of this investigation] may be rewritten as

$$\vec{E}_p^{(+)}(\vec{r},\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{ei}{2\pi^2 v}\,\frac{\left(p,q,\dfrac{v}{\omega}\left[\dfrac{\omega^2}{v^2}-\dfrac{\omega^2}{c^2}\epsilon(\vec{k}_v,\omega)\right]\right)}{\epsilon(\vec{k}_v,\omega)\left(\vec{k}_v^2-\dfrac{\omega^2}{c^2}\epsilon(\vec{k}_v,\omega)\right)}e^{i\vec{k}_v\cdot\vec{r}}\,dp\,dq\,, \tag{2.1}$$

where

$$\vec{k}_v=(p,q,\omega/v)\,,$$

and

$$\epsilon(\vec{k}_v,\omega)=\epsilon_0+\frac{\chi}{\vec{k}_v^{\,2}-\mu^2(\omega)}. \tag{2.2}$$

As pointed out in I we identify (2.1) with the Čerenkov field inside the spatially dispersive half-space. The double integral in (2.1) can be evaluated exactly. For this purpose we introduce the following transformations:

$$p=l\cos\varphi\,,\quad x=\rho\cos\theta\,,$$
$$q=l\sin\varphi\,,\quad y=\rho\sin\theta\,. \tag{2.3}$$

Then (2.1) can be rewritten as

$$\vec{E}_P^{(+)}(\vec{r},\omega)=\int_0^{2\pi}d\varphi\int_0^{\infty}ldl\left(\frac{ei}{2\pi^2 v}\right)\exp\left[i\left(l\rho\cos(\varphi-\theta)+\frac{\omega}{v}z\right)\right]\frac{\left(l\cos\varphi,\,l\sin\varphi,\,\frac{v}{\omega}\left[\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon\left(l^2+\frac{\omega^2}{v^2},\omega\right)\right]\right)}{\epsilon\left(l^2+\frac{\omega^2}{v^2},\omega\right)\left[l^2+\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon\left(l^2+\frac{\omega^2}{v^2},\omega\right)\right]}. \tag{2.4}$$

The integration with respect to φ can be easily performed and (2.4) then reduces to

$$\vec{E}_P^{(+)}(\vec{r},\omega)=\frac{ei}{\pi v}e^{i(\omega/v)z}\int_0^{\infty}ldl\,\frac{\left(l\cos\theta J_1(l\rho),\,l\sin\theta J_1(l\rho),\,\frac{v}{\omega}\left[\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon\left(l^2+\frac{\omega^2}{v^2},\omega\right)\right]J_0(l\rho)\right)}{\epsilon\left(l^2+\frac{\omega^2}{v^2},\omega\right)\left[l^2+\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon\left(l^2+\frac{\omega^2}{v^2},\omega\right)\right]}. \tag{2.5}$$

where J_0 and J_1 are Bessel functions. The integral in (2.5) can be evaluated by using an integral representation for the modified Bessel functions of the third kind. To do this we expand the integrand of (2.5) in terms of partial fractions. Equation (2.5) then becomes

$$\vec{E}_P^{(+)}(\vec{r},\omega)=\frac{ei}{\pi v}e^{i(\omega/v)z}\left(\sum_{j=1}^{3}C_j^{(0)}(\omega)\int_0^{\infty}\frac{l^2dlJ_1(l\rho)}{l^2+\omega^2/v^2-k_j^2}(\hat{x}\cos\theta+\hat{y}\sin\theta)+\sum_{j=1}^{3}C_j^{(1)}(\omega)\int_0^{\infty}\frac{ldlJ_0(l\rho)}{l^2+\omega^2/v^2-k_j^2}\hat{z}\right), \tag{2.6}$$

where

$$C_j^{(0)}(\omega)=\frac{n_j^2-\mu^2/k_0^2}{\prod_{j'\neq j}(n_j^2-n_{j'}^2)},\quad j,j'=1,2,3 \tag{2.7}$$

and

$$C_j^{(1)}(\omega)=\frac{v}{\omega}\left(\frac{\omega^2}{\epsilon_0 v^2}-\frac{\omega^2}{c^2}-\frac{\chi}{\epsilon_0(n_j^2-\mu^2/k_0^2)}\right)C_j^{(0)}(\omega). \tag{2.8}$$

In (2.7) and (2.8) $n_j=k_j/k_0$, and $k_3=k_l$. The integrals appearing in (2.6) are of the form[2]

$$\int_0^{\infty}J_\mu(bt)(t^2+z^2)^{-\nu}t^{\mu+1}dt=(\tfrac{1}{2}b)^{\nu-1}\frac{z^{1+\mu-\nu}}{\Gamma(\nu)}K_{\nu-\mu-1}(bz),\quad \mathrm{Re}(2\nu-\tfrac{1}{2})>\mathrm{Re}\,\mu>-1,\quad \mathrm{Re}\,z>0, \tag{2.9}$$

where K denotes the modified Bessel function of the third kind, and we note that

$$K_{-1}(x)=K_1(x). \tag{2.10}$$

Using Eqs. (2.9)–(2.10), Eq. (2.6) can be written as

$$\vec{E}_P^{(+)}(\vec{r},\omega)=\frac{ei}{\pi v}e^{i(\omega/v)z}\sum_{j=1}^{3}\left(a_j(\omega)C_j^{(0)}(\omega)K_1(a_j\rho)\cos\theta,\,a_j(\omega)C_j^{(0)}(\omega)K_1(a_j\rho)\sin\theta,\,C_j^{(1)}(\omega)K_0(a_j\rho)\right), \tag{2.11}$$

where

$$a_j(\omega)=i\left(k_0^2n_j^2-\frac{\omega^2}{v^2}\right)^{1/2}, \tag{2.12}$$

and

$$\mathrm{Re}\,a_j(\omega)>0.$$

We have now expressed the Čerenkov field $\vec{E}_P^{(+)}$ in a closed form in terms of Bessel functions. This expression is valid for all points inside the spatially dispersive half-space. However, in order to understand the physical nature of the Čerenkov field we will now obtain an asymptotic form for the Čerenkov field away from the path of the electron.

For this purpose we use the asymptotic expansion of the modified Bessel function of the third kind[3] which is

$$K_1(a_j\rho)=K_0(a_j\rho)=\left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-a_j\rho}}{(a_j\rho)^{1/2}}\left[1+O\left(\frac{1}{\rho a_j}\right)\right]. \tag{2.13}$$

For points far away from the path of the electron, i.e., for points such that

$$\max_{(j)}(|a_j|)\rho\gg1, \tag{2.14}$$

the asymptotic form of the Čerenkov field may be obtained if in Eq. (2.12) the asymptotic forms for K_1 and K_0 [Eq. (2.13)] are substituted. We then obtain

$$\vec{E}_P^{(+)}(\vec{r},\omega)=\frac{ei}{\pi v}e^{i(\omega/v)z}\sum_{j=1}^{3}\left(\frac{\pi}{2\rho a_j}\right)^{1/2}(\cos\theta,\sin\theta,a_j^2)a_j(\omega,v)e^{-a_j\rho}\left[1+O\left(\frac{1}{a_j\rho}\right)\right]. \tag{2.15}$$

Using the definition (2.12) of the numbers a_j we deduce, since the a_j's are complex, from (2.15) that the Čerenkov field undergoes damped oscillations with respect to distance ρ. The degree to which the field is damped depends on the real part of a_j which is

$$\mathrm{Re}(a_j)=\frac{1}{\sqrt{2}}\left\{\left[\left(k_0^2\left(n_{jr}^2-n_{ji}^2-\frac{c^2}{v^2}\right)\right)^2+4k_0^4n_{jr}^2n_{ji}^2\right]^{1/2}-k_0^2\left(n_{jr}^2-n_{ji}^2-\frac{c^2}{v^2}\right)\right\}^{1/2}, \tag{2.16}$$

where

$$n_j\equiv n_{jr}+in_{ji}$$

and n_{jr} and n_{ji} are real functions of ω.

From the structure of (2.16) one may readily deduce that $\mathrm{Re}(a_j)$ decreases rapidly for velocities for which

$$n_{jr}^2-n_{ji}^2-\frac{c^2}{v^2}>0. \tag{2.17}$$

In the case of no damping (which means $n_{ji}=0$, $\mathrm{Re}(a_j)$ becomes zero [the minimum value for $\mathrm{Re}(a_j)$] for all values of v satisfying (2.17). Since a small value of $\mathrm{Re}(a_j)$ corresponds to a weak damping of the asymptotic field $\vec{E}_P^{(+)}(\vec{r},\omega)$ [Eq. (2.15)], we identify the velocities given by the equations

$$n_{jr}^2-n_{ji}^2-\frac{c^2}{v^2}=0\quad(j=1,2,3) \tag{2.18}$$

as the threshold velocities. There are in general three threshold velocities corresponding to the two transverse waves and the one longitudinal wave.

We rewrite the threshold conditions in the form

$$v_j=\frac{c}{(n_{jr}^2-n_{ji}^2)^{1/2}},\ \text{if } n_{jr}>n_{ji},\quad j=1,2,3. \tag{2.19}$$

We note that when $n_{jr}<n_{ji}$, it is not possible to satisfy Eq. (2.17). Therefore, there will be no threshold velocities for frequencies for which

$$n_{jr}<n_{ji}. \tag{2.20}$$

Thus, it is possible that in certain cases only one of the three different Čerenkov fields is radiated, whereas the others are strongly damped. For a realistic-model medium, for example the one considered in Ref. 4, the real part of only one of the refractive indices is greater than its imaginary part (as can be deduced from Fig. 2 of Ref. 4). One would therefore conclude that in this case the dominating contribution to the Čerenkov field in the far zone (above threshold) would come from the transverse mode whose refractive index satisfies the condition that

$$n_{jr}>n_{ji}. \tag{2.21}$$

III. TRANSITION RADIATION

In Sec. III of paper I we obtained expressions [Eqs. I(4.12) and I(4.22–I(4.24)] for the amplitudes of the homogeneous part of the electric field outside the medium in the p, q, and ω representation. We identified this part of the field with the transition radiation field. If we take the inverse Fourier transformation of Eqs. I(4.22)–I(4.24) with respect to p and q, we obtain, for each frequency component, the transition radiation field in real space. Unfortunately, unlike in the case of Čerenkov radiation, it does not seem to be possible to evaluate these integrals in a closed form and we will, therefore, have to resort to approximate methods. However, generally one is interested in the behavior of the transition radiation field for points at distances away from the boundary and the path of the electron, which are large compared to the wavelength. We will therefore derive the asymptotic expansion of the transition radiation field at such distances. For the spatially nondispersive case the corresponding expressions were obtained by Garibyan.[5]

We explicitly evaluate the determinants appearing in Eqs. I(4.22) and I(4.23) and then obtain

$$(\vec{E}_{\mathrm{TR}}^{(-)})_x=\Delta^{-1}\{(Q+\alpha w_0)[Q+\delta(p^2+q^2)]\mathcal{E}_x(p,q)+\alpha(Q+\alpha w_0)[1-\beta(p^2+q^2)]\mathcal{E}_x'(p,q)\}, \tag{3.1}$$

$$(\vec{E}_{\mathrm{TR}}^{(-)})_y=\Delta^{-1}\left\{(Q+\alpha w_0)[Q+\delta(p^2+q^2)]\frac{q}{p}\mathcal{E}_x(p,q)+\alpha(Q+\alpha w_0)[1-\beta(p^2+q^2)]\frac{q}{p}\mathcal{E}_x'(p,q)\right\}, \tag{3.2}$$

and

$$(\vec{E}_{\mathrm{TR}}^{(-)})_z=-\frac{p}{m_0}(\vec{E}_{\mathrm{TR}}^{(-)})_x-\frac{q}{m_0}(\vec{E}_{\mathrm{TR}}^{(-)})_y, \tag{3.3}$$

where

$$\Delta = Q^2 + Q\left(\delta(p^2+q^2) - \frac{\alpha\beta}{w_0}(p^2+q^2)^2 + \frac{\alpha}{w_0}(p^2+q^2) + 2\alpha w_0 - \alpha\beta w_0(p^2+q^2)\right) + \alpha\delta(p^2+q^2)w_0 + \alpha^2 k_0^2[1-(p^2+q^2)\beta]\,. \tag{3.4}$$

The quantities Q, α, β, γ, and δ are defined as before, viz.,

$$\alpha = \frac{w_1 - w_2}{w_\mu - w_2}, \quad \beta = \frac{w_3 - w_1}{(\vec{k}_1 \cdot \vec{k}_3) w_2}, \quad \gamma = w_\mu - w_1,$$

$$\delta = \alpha\beta\gamma + \frac{\alpha}{w_2}, \quad Q = w_1(1+\alpha) - w_2\,. \tag{3.5}$$

where w_1 and w_2 are the roots of the dispersion equation I(3.17), i.e., of the equation

$$\chi k_0^2 - (\mu^2 - p^2 - q^2 - w^2)(\mu^2 - p^2 - q^2 - w^2 + k_0^2\epsilon_0) = 0\,, \tag{3.6}$$

with positive real part, and w_3 is the root, with positive real part, of the dispersion relation I(3.18), i.e., of the equation

$$\mu^2 - p^2 - q^2 - w^2 - \chi/\epsilon_0 = 0\,. \tag{3.7}$$

The functions $\mathcal{E}_x$ and $\mathcal{E}_x'$ are given by the expressions Eqs. I(4.29) and I(4.30). After substituting the expressions I(4.15) of paper I for $\vec{E}_P^{(+)}$ and I(4.6) for $\vec{E}_c^{(-)}$ into Eqs. I(4.29) and I(4.30) a lengthy but straightforward calculation yields

$$\mathcal{E}_x(p,q) = \frac{eip}{2\pi^2 v}\left(\frac{w_1 - \omega/v}{w_\mu - \omega/v}\{\epsilon(\vec{k}_v,\omega)[\vec{k}_v^2 - k_0^2\epsilon(\vec{k}_v,\omega)]\}^{-1} \right.$$
$$\left. + \frac{w_3 - w_1}{w_\mu - \omega/v}\,\frac{p^2+q^2+w_1\dfrac{v}{\omega}\left(\dfrac{\omega^2}{v^2} - \dfrac{\omega^2}{c^2}\epsilon(\vec{k}_v,\omega)\right)}{\epsilon(\vec{k}_v,\omega)[\vec{k}_v^2 - k_0^2\epsilon(\vec{k}_v,\omega)](p^2+q^2+w_1 w_3)} - \frac{1}{\vec{k}_v^2 - k_0^2}\right) \equiv pF(p^2+q^2, v, \omega)\,, \tag{3.8}$$

and

$$\mathcal{E}_x'(p,q) = \frac{eip}{2\pi^2 v}\left\{\frac{w_\mu - w_1}{w_\mu - \omega/v}\left(\frac{w_3 - w_1}{w_\mu - \omega/v}\,\frac{p^2+q^2+w_1\dfrac{v}{\omega}\left(\dfrac{\omega^2}{v^2} - \dfrac{\omega^2}{c^2}\epsilon(\vec{k}_v,\omega)\right)}{\epsilon(\vec{k}_v,\omega)[\vec{k}_v^2 - k_0^2\epsilon(\vec{k}_v,\omega)](p^2+q^2+w_1)} - \frac{\dfrac{v}{\omega}\left(\dfrac{\omega^2}{v^2} - \dfrac{\omega^2}{c^2}\epsilon(\vec{k}_v,\omega)\right) + w_1}{\epsilon(\vec{k}_v,\omega)[\vec{k}_v^2 - k_0^2\epsilon(\vec{k}_v,\omega)]}\right)\right.$$
$$+ \frac{\omega}{v}\,\frac{1 - v^2/c^2}{\vec{k}_v^2 - k_0^2} - \frac{\dfrac{\omega}{v}\left(1 - \dfrac{v^2}{c^2}\epsilon(\vec{k}_v,\omega)\right)}{\epsilon(\vec{k}_v,\omega)[\vec{k}_v^2 - k_0^2\epsilon(\vec{k}_v,\omega)]}$$
$$\left. + \frac{\omega}{v}\left(\frac{1}{\epsilon(\vec{k}_v,\omega)[\vec{k}_v^2 - k_0^2\epsilon(\vec{k}_v,\omega)]} - \frac{1}{\vec{k}_v^2 - k_0^2}\right)\right\} \equiv pG(p^2+q^2, v, \omega)\,. \tag{3.9}$$

Using Eqs. (3.4), (3.8), and (3.9) in Eqs. (3.1)–(3.3) we obtain the following expression for the transition radiation field:

$$\vec{E}_{TR}^{(-)} = \left(p, q, -\frac{p^2+q^2}{m_0}\right) T(p^2+q^2, v, \omega)\,, \tag{3.10}$$

where

$$T(p^2+q^2, v, \omega) = \Delta^{-1}\{(Q+\alpha w_0)[Q+\delta(p^2+q^2)]\,F(p^2+q^2, v, \omega) + \alpha(Q+\alpha w_0)[1-\beta(p^2+q^2)]G(p^2+q^2, v, \omega)\}\,. \tag{3.11}$$

Hence, Eqs. I(3.10) and I(3.4) of paper I yield, for each frequency component, the following expression for the transition radiation field outside the spatial dispersive half space:

$$\vec{E}^{(-)}_{TR}(\vec{r},\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}T(p^2+q^2,v,\omega)\left(p,q,-\frac{p^2+q^2}{m_0}\right)\exp[ipx+iqy-i(k_0^2-p^2-q^2)^{1/2}z]dpdq\,, \tag{3.12}$$

where

$$\mathrm{Im}(k_0^2-p^2-q^2)^{1/2}>0\,,\ \text{for } p^2+q^2>k_0^2\,.$$

As can be seen from (3.11) the integrand appearing in the double integral (3.12) is a complicated function of p and q and it does not seem to be possible to evaluate it explicitly. However, the double integral can be transformed into a single integral by introduction of the cylindrical coordinate system

$$p=l\cos\varphi\,,\quad q=l\sin\varphi\,,\quad x=\rho\cos\theta,\ \ y=\rho\sin\theta, \tag{3.13}$$

and by performing the angular integration as we did before in the case of Čerenkov radiation in Sec. II. Then (3.9) becomes

$$\vec{E}^{(-)}_{TR}(\vec{r},\omega)=\int_0^{\infty}l^2dl\,T(l^2,v,\omega)\exp[-i(k_0^2-l^2)^{1/2}z]\left(\cos\theta J_1(l\rho),\ \sin\theta J_1(l\rho),\frac{l}{(k_0^2-l^2)^{1/2}}J_0(l\rho)\right). \tag{3.14}$$

We will now give the asymptotic expansion of the field for points many wavelengths away from both the boundary and the path of the electron. (The derivation of these expansions is given in the Appendix.)

$$(\vec{E}^{(-)}_{TR})_x\sim\cos\theta\,\mathcal{A}(v,\omega,\alpha)\frac{\exp(+ik_0R)}{R}+O\left(\frac{1}{(k_0R)^2}\right), \tag{3.15}$$

$$(\vec{E}^{(-)}_{TR})_y\sim\sin\theta\,\mathcal{A}(v,\omega,\alpha)\frac{\exp(+ik_0R)}{R}+O\left(\frac{1}{(k_0R)^2}\right), \tag{3.16}$$

$$(\vec{E}^{(-)}_{TR})_z\sim\mathcal{B}(v,\omega,\alpha)\frac{\exp(+ik_0R)}{R}+O\left(\frac{1}{(k_0R)^2}\right), \tag{3.17}$$

where

$$\mathcal{A}(v,\omega,\alpha)=\exp(\tfrac{3}{4}\pi i)i^{1/2}(2\pi)^{-1/2}(\tfrac{1}{2}\pi)^{1/2}\times k_0\sin\alpha\cos\alpha\,T(-k_0,v,\omega)\,, \tag{3.18}$$

and

$$\mathcal{B}(v,\omega,\alpha)=-i\cot\alpha\,\mathcal{A}(v,\omega,\alpha)\,. \tag{3.19}$$

The angle α is defined by

$$\rho=R\cos\alpha\,,\quad z=-R\sin\alpha\,, \tag{3.20a}$$

$$R=(x^2+y^2+z^2)^{1/2}\,. \tag{3.20b}$$

Equations (3.15)–(3.17) reveal that, asymptotically, the transition radiation in vacuum consists of spherical waves. Moreover, recalling that θ is the angle between the x axis and ρ, Eqs. (3.15)–(3.17) show that the electric field vector of the transition radiation lies in the plane containing the line joining to the point of observation to the point where the charged particle entered the medium and the trajectory of the particle. For the spatially nondispersive case, this asymptotic property of the transition radiation field was already observed by Garibyan.[5]

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APPENDIX: DERIVATION OF THE ASYMPTOTIC EXPANSION (3.15)–(3.17)

In order to derive the asymptotic expansion (3.15)–(3.17) we substitute the asymptotic expansions for Bessel functions, viz.

$$J_0(l\rho)=\left(\frac{2}{\pi l\rho}\right)^{1/2}\cos(l\rho-\tfrac{1}{4}\pi)\left[1+O\left(\frac{1}{l\rho}\right)\right] \tag{A1}$$

and

$$J_1(l\rho)=\left(\frac{2}{\pi l\rho}\right)^{1/2}\cos(l\rho-\tfrac{3}{4}\pi)\left[1+O\left(\frac{1}{l\rho}\right)\right], \tag{A2}$$

into (3.14). We then obtain the expressions

$$\vec{E}^{(-)}_{TR}(\vec{r},\omega)=\int_0^{\infty}l^2dl\left(\frac{2}{\pi l\rho}\right)^{1/2}\left(\cos\theta\cos(l\rho-\tfrac{3}{4}\pi)\,,\ \sin\theta\cos(l\rho-\tfrac{3}{4}\pi)\,,\ -\frac{l}{(k_0^2-l^2)^{1/2}}\cos(l\rho-\tfrac{1}{4}\pi)\right)\times T(l^2,v,\omega)\exp[-i(k_0^2-l^2)^{1/2}z]\,. \tag{A3}$$

Strictly speaking, near the point $l=0$, (A1), (A2), and therefore (A3) are no longer valid. It was however shown by Hilb,[6] that by the transition from (3.14) to (A3) one introduces an error of the order $O(1/k_0^2\rho^2)$ which is already of lower order than the order of the leading term of the asymptotic expansion to be established.

We will now evaluate a typical integral which appears in (A3) for large values of both ρ and z. Making the substitutions

$$\rho = R\cos\alpha, \quad z = -R\sin\alpha, \tag{A4}$$

the $\hat{x}$ component of (A3) can be rewritten as

$$\cos\theta\left(\frac{1}{2\pi R\cos\alpha}\right)^{1/2}\int_0^\infty l^{3/2}dl\,T(l^2,v,\omega)\left\{\exp\left[iR\left(l\cos\alpha-\frac{3\pi}{4R}-(k_0^2-l^2)^{1/2}\sin\alpha\right)\right]\right.$$
$$\left.+\exp\left[iR\left(-l\cos\alpha+\frac{3\pi}{4R}-(k_0^2-l^2)^{1/2}\sin\alpha\right)\right]\right\}. \tag{A5}$$

Let us define two functions $T_1(R,\alpha,v,\omega)$ and $T_2(R,\alpha,v,\omega)$ by

$$T_1(R,\alpha,v,\omega)=\int_0^\infty l^{3/2}dl\,T(l^2,v,\omega)\exp\{iR[-l\cos\alpha-(k_0^2-l^2)^{1/2}\sin\alpha]\}, \tag{A6}$$

$$T_2(R,\alpha,v,\omega)=\int_0^\infty l^{3/2}dl\,T(l^2,v,\omega)\exp\{iR[l\cos\alpha-(k_0^2-l^2)^{1/2}\sin\alpha]\}. \tag{A7}$$

We will now obtain the asymptotic expansion of (A6). For this purpose we introduce a new variable $y(l)$ defined by

$$y(l,\alpha)=-l\cos\alpha-(k_0^2-l^2)^{1/2}\sin\alpha, \tag{A8}$$

and note that according to (3.12) $\mathrm{Im}(k_0^2-l^2)^{1/2}>0$ if $l>k_0$. Let L_1 denote the mapping of the interval $0\le l<k_0\cos\alpha$ under the transformation (A8), i.e.,

$$L_1=\{y:+k_0\ge y\ge +k_0\sin\alpha\}, \tag{A9}$$

and L_2 the mapping of the interval $k_0\cos\alpha\le l<\infty$ under the transformation (A8). Then one has

$$l(y)=+y\cos\alpha-\sin\alpha\,(k_0^2-y^2)^{1/2}\text{ if } y\in L_1,$$
$$=+y\cos\alpha+\sin\alpha\,(k_0^2-y^2)^{1/2}\text{ if } y\in L_2. \tag{A10}$$

Changing the variable l in (A6) to y yields

$$T_1(R,\alpha,v,\omega)=\int_{L_1+L_2} l^{3/2}(y)dy\,T(y,v,\omega)\times\exp(iyR)\left(\frac{dl}{dy}\right), \tag{A11}$$

where

$$\frac{dl}{dy}=+\cos\alpha+\sin\alpha\,(k_0^2-y^2)^{-1/2}y\text{ if } y\in L_1,$$
$$=+\cos\alpha-\sin\alpha\,(k_0^2-y^2)^{-1/2}y\text{ if } y\in L_2. \tag{A12}$$

The function (dl/dy) has one term in common if $y\in L_1$ or $y\in L_2$, namely, $+\cos\alpha$. The contribution of this term to the asymptotic expansion of (A11) can be easily calculated. Using the identity

$$\exp(iyR)\equiv\frac{1}{iR}\frac{d}{dy}[\exp(iyR)], \tag{A13}$$

integration by parts yields

$$\int_{L_1+L_2} l^{3/2}(y)dy\,T(y,v,\omega)\exp(iyR)(-\cos\alpha)=l^{3/2}(y)\,T(y,v,\omega)(-\cos\alpha)\frac{\exp(iyR)}{iR}\Bigg|_{+k_0\sin\alpha}^{+k_0+i\infty}$$
$$+\frac{1}{iR}\int_{L_1+L_2}\frac{d}{dy}[l^{3/2}(y)T(y,v,\omega)]\exp(iyR)(+\cos\alpha)dy=O\left(\frac{1}{k_0R}\right). \tag{A14}$$

We will now calculate the contribution to the asymptotic expansion of (A11) of the second term in the right-hand side of (A12). Using Cauchy's theorem along a rectangle with vertices $(-k_0\sin\alpha, +k_0\sin\alpha+i\infty, +k_0+i\infty, +k_0)$, we find that

$$\int_{L_1} l^{3/2}(y)T(y,v,\omega)\exp(iyR)\sin\alpha(k_0^2-y^2)^{-1/2}y\,dy$$
$$=\sum_j \int_{c_j} l^{3/2}(y)T(y,v,\omega)\exp(iyR)\sin\alpha(k_0^2-y^2)^{-1/2}y\,dy$$
$$+\int_{\epsilon>0}^{\infty} l^{3/2}(-k_0+i\tau)T(+k_0+i\tau,v,\omega)\exp[iR(+k_0+i\tau)](-i\tau)^{-1/2}(2k_0+i\tau)^{-1/2}\sin\alpha\, i\,d\tau$$
$$-\int_0^{\infty} l^{3/2}(+k_0\sin\alpha+i\tau)T(+k_0\sin\alpha+i\tau,v,\omega)\exp[iR(+k_0\sin\alpha+i\tau)][k_0^2-(+k_0\sin\alpha+i\tau)^2]^{-1/2}$$
$$\times(+k_0\sin\alpha+i\tau)\sin\alpha\, i\,d\tau\,. \tag{A15}$$

The integral over the contour from $+k_0\sin\alpha+i\infty$ to $+k_0+i\infty$ is zero, since this integral decays exponentially to zero if $\mathrm{Im}\,y\to\infty$. The contours c_j are around the singular points of the function $T(l,v,\omega)$ within the rectangle, which by assumption have a nonvanishing imaginary part. Hence,

$$\int_{c_j} l^{3/2}(y)T(y,v,\omega)\exp(iyR)\sin\alpha(k_0^2-y^2)^{-1/2}y\,dy=O\left(\frac{1}{k_0R}\right). \tag{A16}$$

By an analysis similar to the one leading to (A14), we obtain the order of the asymptotic expansion of the last term of the right-hand side of (A15):

$$\int_0^{\infty} l^{3/2}(+k_0\sin\alpha+i\tau)T(+k_0\sin\alpha+i\tau,v,\omega)\exp[iR(+k_0\sin\alpha+i\tau)]$$
$$\times[k_0^2-(+k_0\sin\alpha+i\tau)^2]^{-1/2}(+k_0\sin\alpha+i\tau)\sin\alpha\, i\,d\tau=O\left(\frac{1}{k_0R}\right). \tag{A17}$$

The second term of the right-hand side of (A15) will give us the leading term in the asymptotic expansion of (A11). Substituting $\tau R=q$ into this term we obtain, because the exponential in the integrand is independent of R,

$$\int_0^{\infty} l^{3/2}(+k_0+i\tau)T(+k_0+i\tau,v,\omega)\sin\alpha\exp[i(+k_0+i\tau)R](i\tau)^{-1/2}(2k_0+i\tau)^{-1/2}i\,d\tau$$
$$=\frac{\exp(+ik_0R)}{R^{1/2}}\int_0^{\infty} l^{3/2}\left(k_0+\frac{iq}{R}\right)T\left(+k_0+\frac{iq}{R},v,\omega\right)\sin\alpha\exp(-q)i^{1/2}q^{-1/2}\left(2k_0+\frac{iq}{R}\right)^{-1/2}q^{-1/2}dq$$
$$=\frac{\exp(+ik_0R)}{R^{1/2}}\int_0^{\infty} l^{3/2}(+k_0)T(+k_0,v,\omega)\sin\alpha\exp(-q)i^{1/2}q^{-1/2}(2k_0)^{-1/2}dq\left[1+O\left(\frac{1}{k_0R}\right)\right]. \tag{A18}$$

Combination of (A15)–(A18) leads to

$$\int_{L_1} l^{3/2}(y)T(y,v,\omega)\exp(iyR)\sin\alpha(k_0^2-y^2)^{-1/2}y\,dy=\mathfrak{A}'(v,\omega,\alpha)\frac{\exp(+ik_0R)}{R^{1/2}}+O\left(\frac{1}{k_0R}\right), \tag{A19}$$

where

$$\mathfrak{A}'(v,\omega,\alpha)=\int_0^{\infty} l^{3/2}(+k_0)T(+k_0,v,\omega)\sin\alpha[\exp(-q)i^{1/2}(2k_0q)^{-1/2}]\,dq$$
$$=i^{1/2}\left(\frac{\pi}{2}\right)^{1/2}k_0\sin\alpha\cos^{3/2}\alpha\, T(+k_0,v,\omega)\,. \tag{A20}$$

Similarly, we find that

$$\int_{L_2} l^{3/2}(y)T(y,v,\omega)\exp(iyR)(-\sin\alpha)(k_0^2-y^2)^{-1/2}y\,dy$$
$$=\int_0^{\infty} l^{3/2}(+k_0+i\tau)T(+k_0+i\tau,v,\omega)\exp[iR(+k_0+i\tau)]\sin\alpha(i\tau)^{-1/2}(2k_0+i\tau)^{-1/2}i\,d\tau+O\left(\frac{1}{k_0R}\right), \tag{A21}$$

from which, on setting $\tau R=q$, we obtain the asymptotic approximation

$$\int_{L_2} l^{3/2}(y)T(y,v,\omega)\exp(iyR)(-\sin\alpha)(k_0^2-y^2)^{-1/2}y\,dy=\frac{\exp(+ik_0R)}{R^{1/2}}\mathfrak{a}'(v,\omega,\alpha)+O\left(\frac{1}{k_0R}\right). \tag{A22}$$

Combination of (A6), (A11), (A19), and (A22) yields[7]

$$T_1(R,\alpha,v,\omega)=\mathfrak{a}'(v,\omega,\alpha)\frac{\exp(+ik_0R)}{R^{1/2}}+O\left(\frac{1}{k_0R}\right). \tag{A23}$$

We shall now derive the asymptotic expansion of (A6). Let

$$x=-l\cos\alpha+(k_0^2-l^2)^{1/2}\sin\alpha,\ \text{if } 0\le l<\infty,$$

and (A24)

$$\mathrm{Im}(k_0^2-l^2)^{1/2}>0 \text{ if } l>k_0.$$

Then

$$l=-x\cos\alpha+(k_0^2-x^2)^{1/2}\sin\alpha. \tag{A25}$$

A change of the variable from l to x in (A7) yields

$$T_2(R_1,\alpha,v,\omega)=\int_C l^{3/2}(x)T(x,v,\omega)\times\exp(ixR)\left(\frac{dl}{dx}\right)dx, \tag{A26}$$

where the contour C is the mapping of the interval $[0,\infty]$ of the real l axis onto the complex x plane under the transformation (A24).

It is readily observed from (A25) that both dl/dx and d^2l/dx^2 exist everywhere on the contour C. Integration by parts of (A26) using (A13) yields

$$T_2(R,\alpha,v,\omega)=O\left(\frac{1}{k_0R}\right). \tag{A27}$$

Using (A24) and (A27) we obtain from (A5)–(A7) the aymptotic expansion of $(\vec{E}_{TR}^{(-)})_x$ as

$$(\vec{E}_{TR}^{(-)})_x=\cos\theta\exp(\tfrac{3}{4}\pi i)(2\pi\cos\alpha)^{-1/2}\mathfrak{a}'(v,\omega,\alpha)\times[\exp(+ik_0R)/R]\left[1+O\left(\frac{1}{k_0R}\right)\right]. \tag{A28}$$

Similarly, we obtain

$$(\vec{E}_{TR}^{(-)})_y=\sin\theta\exp(\tfrac{3}{4}\pi i)(2\pi\cos\alpha)^{-1/2}\mathfrak{a}'(v,\omega,\alpha)\times[\exp(+ik_0R)/R]\left[1+O\left(\frac{1}{k_0R}\right)\right] \tag{A29}$$

and

$$(\vec{E}_{TR}^{(-)})_z=\exp(\tfrac{3}{4}\pi i)(2\pi\cos\alpha)^{-1/2}(-\cot\alpha)\mathfrak{a}'(v,\omega,\alpha)\times[\exp(+ik_0R)/R]\left[1+O\left(\frac{1}{k_0R}\right)\right]. \tag{A30}$$

The formulas (A28)–(A30) can easily be written in the form (3.15)–(3.17), respectively, if one uses Eq. (A20).

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[7]It might seem remarkable that on adding the leading terms of the asymptotic expansions of (A19) and (A22) they do not cancel, for both the leading terms come from the contribution of the point $y=k_0$, which is approached from the right in (A19) and from the left in (A22). However, they add because of the change in sign of the second term at the right-hand side of the function $l(y)$, Eq. (A10).